Hamiltonian formulation of magnetic field line equations

1987 J. Phys. A: Math. Gen. 20 3679

(http://iopscience.iop.org/0305-4470/20/12/021)

J. Phys. A: Math. Gen. **20** (1987) 3679-3685. Printed in the UK

Hamiltonian formulation of magnetic field line equations

M Sita Janaki and Gautam Ghosh

Saha Institute of Nuclear Physics, 92 APC Road, Calcutta 700 009, India

Received 24 June 1986

Abstract. A Hamiltonian formulation of magnetic field line equations is derived for arbitrary magnetic field configurations in orthogonal curvilinear coordinate systems. The canonical equivalence of the different descriptions thus obtained are explicitly demonstrated and action-angle forms are given in cases where there is a geometrical symmetry. Examples of Hamiltonians are discussed for usual plasma machines like the tokamak and the levitron. Finally generalisation to non-orthogonal coordinate systems is worked out.

1. Introduction

Magnetic coordinate systems (Boozer 1983, Carey and Littlejohn 1983) have been used in plasma physics for quite some time. These are the natural coordinates that can be employed to describe the geometry of magnetic surfaces in fusion machines (see, for example, Boozer 1985a, b, Miyamoto 1985). But the use of poloidal and toroidal quantities from the very beginning ties up the procedure to particular field configurations (Channel 1984) and obscures the generality of the Hamiltonian description that can be obtained independent of the geometry of field lines or the particular coordinate system used. In fact a time evolution under Hamiltonian flow and the equations for the lines of magnetic force both exhibit certain conserved quantities. It is the formal analogy between the respective conserved quantities that is exploited in the following to construct a Hamiltonian description in a very general way.

The motivation for such work is manyfold and stems primarily from the fact that once a Hamiltonian description is achieved the motion of charged particles along the field lines becomes constrained to the usual phase space geometries allowed by the corresponding Hamiltonian. In particular if there is a geometrical symmetry in the magnetic field (say, invariance under translation in a particular direction) the Hamiltonian becomes time independent and since this is a one degree of freedom system, it is trivially integrable. Thus particles starting out on a given magnetic surface characterised by a particular value of the Hamiltonian can never leave it. Of greater interest are the quasi-integrable systems where a small symmetry breaking perturbation is present (for example, an oscillating magnetic field) and it becomes of practical importance (in fusion machines) to study the onset of chaos through the successive breaking of magnetic surfaces into chains of islands (Freis *et al* 1973, Rechester *et al* 1979, Lichtenberg and Lieberman 1983). The techniques of Hamiltonian dynamics like canonical perturbation theory, discrete mappings, KAM stability, etc (Arnold and Avez 1968, Arnold 1979, Henon 1983) are then the essential tools of study.

0305-4470/87/123679+07\$02.50

2. Hamiltonian formulation

Let us adopt an orthogonal curvilinear coordinate system (u_1, u_2, u_3) and let the components of a line element $\mathrm{d}s$ be $h_1\,\mathrm{d}u_1$, $h_2\,\mathrm{d}u_2$ and $h_3\,\mathrm{d}u_3$. The lines of force are by definition curves whose tangents give the direction of the magnetic field $\boldsymbol{B}$ which means that the infinitesimal element $\mathrm{d}s$ on the line of force is parallel to $\boldsymbol{B}$. The components of these two vectors are therefore proportional

$$\frac{h_1\,\mathrm{d}u_1}{B_1}=\frac{h_2\,\mathrm{d}u_2}{B_2}=\frac{h_3\,\mathrm{d}u_3}{B_3}. \tag{1}$$

Since $\nabla\cdot\boldsymbol{B}=0$, the lines of force are closed (no singularity) and equation (1) is globally valid. We want to put these equations into a Hamiltonian form. For this purpose we construct canonical coordinate q and momentum p as functions of u_1 and u_2 and single out u_3 to be used as the time variable τ. The derivations that follow do not depend on this choice and in fact we could choose any function of one of the coordinates as the time and independent functions of the other two coordinates as q and p. Thus we have

$$\begin{aligned} q&\equiv q(u_1, u_2)\\ p&\equiv p(u_1, u_2)\\ \tau&\equiv u_3 \end{aligned} \tag{2}$$

and furthermore we demand the existence of a locally single-valued function $H(q, p)$ such that the equations

$$\frac{\partial H}{\partial p}=\frac{\mathrm{d}q}{\mathrm{d}\tau} \qquad \frac{\partial H}{\partial q}=-\frac{\mathrm{d}p}{\mathrm{d}\tau} \tag{3a, b}$$

when expressed in the original coordinates (u_1, u_2, u_3) become identical to the equations for the field lines (1).

Moreover the equality of mixed derivatives

$$\left(\frac{\partial^2 H}{\partial p\,\partial q}=\frac{\partial^2 H}{\partial q\,\partial p}\right)$$

which ensures that H is locally single valued, puts a further restriction on acceptable canonical coordinate and momentum,

$$\frac{\partial}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}\tau}+\frac{\partial}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}\tau}=0. \tag{4}$$

We will see presently that both equation (4) and the demand that equations ($3a, b$) are identical to equation (1) can be satisfied provided that we choose the Jacobian of the transformation from (u_1, u_2) to (q, p) in such a way that the law of flux conservation in the system of equations (1) is automatically carried into the law of volume conservation in phase space in the Hamiltonian system (2) by the mapping itself.

We have

$$\mathrm{d}p\,\mathrm{d}q=J\,\mathrm{d}u_1\,\mathrm{d}u_2 \tag{5}$$

where

$$J=\det\begin{pmatrix}\mathrm{d}q/\mathrm{d}u_1 & \mathrm{d}p/\mathrm{d}u_1\\ \mathrm{d}q/\mathrm{d}u_2 & \mathrm{d}p/\mathrm{d}u_2\end{pmatrix}\equiv\det\frac{\partial(q, p)}{\partial(u_1, u_2)}.$$

The left-hand side of equation (5) is invariant under time evolution since time evolution can be looked upon as a succession of infinitesimal canonical transformations and each canonical transformation has Jacobian unity. The identification of u_3 as the time variable suggests that we look for a corresponding invariance under translation in the u_3 direction. If we choose u_3 to be along the direction of the magnetic field the flux passing through any section of a small tube (figure 1) oriented along the u_3 direction is conserved under translation in u_3.

Thus, $h_1 h_2 \, \mathrm{d}u_1 \, \mathrm{d}u_2 \, B_3$ is conserved and we see that the choice

$$J = \frac{\partial(q, p)}{\partial(u_1, u_2)} = h_1 h_2 B_3 \tag{6}$$

will reduce both sides of (5) to invariant quantities. The condition that the mapping be invertible imposes further conditions on the partial derivatives

$$\begin{pmatrix} \mathrm{d}q \\ \mathrm{d}p \end{pmatrix} = \frac{\partial(q, p)}{\partial(u_1, u_2)} \begin{pmatrix} \mathrm{d}u_1 \\ \mathrm{d}u_2 \end{pmatrix} = \frac{\partial(q, p)}{\partial(u_1, u_2)} \frac{\partial(u_1, u_2)}{\partial(q, p)} \begin{pmatrix} \mathrm{d}q \\ \mathrm{d}p \end{pmatrix}$$

implies

$$\frac{\partial(q, p)}{\partial(u_1, u_2)} \frac{\partial(u_1, u_2)}{\partial(q, p)} = \mathbb{1}. \tag{7}$$

Multiplying by the inverse of one of the matrices we find,

$$\begin{aligned} \frac{\partial q}{\partial u_1} &= J \frac{\partial u_2}{\partial p} & \qquad \frac{\partial q}{\partial u_2} &= -J \frac{\partial u_1}{\partial p} \\ \frac{\partial p}{\partial u_1} &= -J \frac{\partial u_2}{\partial q} & \qquad \frac{\partial p}{\partial u_2} &= J \frac{\partial u_1}{\partial q}. \end{aligned} \tag{8a–d}$$

In equations $(3a, b)$ we now make use of the following equalities:

$$\frac{\partial}{\partial p} = \frac{\mathrm{d}u_1}{\mathrm{d}p} \frac{\partial}{\partial u_1} + \frac{\mathrm{d}u_2}{\mathrm{d}p} \frac{\partial}{\partial u_2}$$

$$\frac{\partial}{\partial q} = \frac{\mathrm{d}u_1}{\mathrm{d}q} \frac{\partial}{\partial u_1} + \frac{\mathrm{d}u_2}{\mathrm{d}q} \frac{\partial}{\partial u_2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\partial}{\partial u_3} + \frac{\mathrm{d}u_1}{\mathrm{d}u_3} \frac{\partial}{\partial u_1} + \frac{\mathrm{d}u_2}{\mathrm{d}u_3} \frac{\partial}{\partial u_2}$$

and make use of equations $(8a–d)$ and (1) to arrive at

$$\{q, H\}_{u_1 u_2} = h_1 h_2 h_3 \boldsymbol{B} \cdot \boldsymbol{\nabla} q \tag{9a}$$

$$\{p, H\}_{u_1 u_2} = h_1 h_2 h_3 \boldsymbol{B} \cdot \boldsymbol{\nabla} p \tag{9b}$$

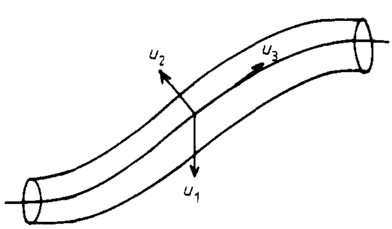

Figure 1. A small flux tube together with the local orthogonal coordinate system used.

where the curly brackets stand for Poisson brackets:

$$\{A, B\}_{u_1 u_2} \equiv \frac{\partial A}{\partial u_1}\frac{\partial B}{\partial u_2} - \frac{\partial A}{\partial u_2}\frac{\partial B}{\partial u_1}.$$

Equations (9*a*, *b*) together with (6) determine the possible structures of q, p and the Hamiltonian H as functions of u_1, u_2 and u_3. Let us analyse them in the following simple cases and examine their canonical equivalence.

3. Canonical equivalence

The structure of the Jacobian matrix shows that if we choose $q(p)$ to be any of u_1 and u_2 the determinant reduces to a single term and then integration yields $p(q)$. Substitution in equation (9*a*, *b*) then gives us the Hamiltonian. Thus we have the following four possibilities:

I $$q_1 = u_1 \qquad p_1 = \int^{u_2} h_1 h_2 B_3 \, \mathrm{d}u_2$$

$$H_1 = \int^{u_2} h_2 h_3 B_1 \, \mathrm{d}u_2$$

II $$q_2 = u_2 \qquad p_2 = -\int^{u_1} h_1 h_2 B_3 \, \mathrm{d}u_1$$

$$H_2 = -\int^{u_1} h_1 h_3 B_2 \, \mathrm{d}u_1$$

III $$p_3 = u_1 \qquad q_3 = -\int^{u_2} h_1 h_2 B_3 \, \mathrm{d}u_2$$

$$H_3 = \int^{u_2} h_2 h_3 B_1 \, \mathrm{d}u_2$$

IV $$p_4 = u_2 \qquad q_4 = \int^{u_1} h_1 h_2 B_3 \, \mathrm{d}u_1$$

$$H_4 = -\int^{u_1} h_1 h_3 B_2 \, \mathrm{d}u_1.$$

Inspection of the above equations reveal that the pair of equations I and III are related by the trivial canonical transformation

$$q_1 \rightarrow p_2$$

$$p_1 \rightarrow -q_2$$

with the generating function $S = -q_1 q_2$.

The pair of equations II and IV are related in a similar manner. It is interesting to find the generating function which takes us from I to II. Let us assume S to be a function of q_1 and p_2 and make a Legendre transformation to a new generating function $F = -p_2 q_2 + S$. Standard manipulations (Goldstein 1980) then yield

$$p_1 = \frac{\partial s}{\partial q_1} \qquad q_2 = \frac{\partial s}{\partial p_2}.$$

An S satisfying the above is obviously

$$S=\int^{u_1}\int^{u_2} h_1 h_2 B_3 \, \mathrm{d}u_1 \, \mathrm{d}u_2 \tag{10}$$

and this checks with the relation

$$H_2 = H_1 + \frac{\partial s}{\partial \tau} \equiv H_1 + \frac{\partial s}{\partial u_3}$$

if we use the relation $\nabla \cdot \boldsymbol{B} = 0$, i.e.

$$\frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}(h_2 h_3 B_1) + \frac{\partial}{\partial u_2}(h_1 h_3 B_2) + \frac{\partial}{\partial u_3}(h_1 h_2 B_3)\right) = 0.$$

4. Action–angle variables

If the magnetic field configuration has a translational symmetry along u_3 then the corresponding Hamiltonian will be time translation invariant and the system reduces to a one degree of freedom conservative system which is integrable. It is possible to transform to action-angle variables in these cases so that the new Hamiltonian is a function of the conserved momentum only. Let us illustrate this with an example of a toroidal configuration of magnetic field lines.

Adopting a toroidal coordinate system (figure 2) and the corresponding line elements $\mathrm{d}r$, $r\,\mathrm{d}\phi$, $(R_0 + r\cos\phi)\,\mathrm{d}\psi$ the magnetic field in a tokamak machine can be written as (Solov'ev and Shafranov 1970)

$$\boldsymbol{B} \equiv \left(0, \frac{B_\phi(r)}{h_s}, \frac{B_0}{h_s}\right) \tag{11}$$

where $h_s = 1 + (r/R_0)\cos\phi$. For a machine of large aspect ratio (large major to minor radius ratio) $h_s \sim 1$ and henceforth we will consider this case only.

We observe that choice II is already in action-angle form,

$$q = \phi \qquad p = -\int^r B_\psi \, \mathrm{d}r = -B_0 r^2/2$$

$$H = -\int^r R_0 B_\phi(r)\,\mathrm{d}r.$$

The Hamiltonian H is independent of ϕ and hence of q. We regain the standard expression (Lichtenberg and Lieberman 1983) for H by introducing the canonical

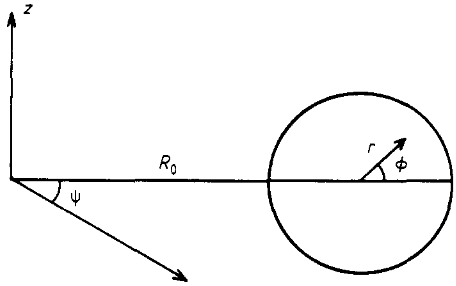

Figure 2. Toroidal coordinate system.

action variable $J(=\int p\,\mathrm{d}q)$ and the rotational transform $\tau(J)(=\int_0^{2\pi}(\mathrm{d}\phi/\mathrm{d}\psi)\,\mathrm{d}\psi)$

$$H=\int^J \tau(J)\,\mathrm{d}J. \tag{12}$$

Similar considerations applied to a levitron field (Freis *et al* 1973)

$$\boldsymbol{B}\equiv\left(B_\nu \sin\phi, \frac{B_0}{\beta r}+B_\nu\cos\phi, B_0\right)$$

where B_ν, β and B_0 are constants, yield the Hamiltonian

$$H=-\frac{B_0R_0}{\beta}\ln r - rB_0B_\nu\cos\phi \tag{13}$$

which can again be written as a series in the canonical action $J(=\int p\,\mathrm{d}q)$.

5. Extension to non-orthogonal coordinate systems

Sometimes it is useful to describe field configurations in terms of non-orthogonal coordinate systems (Morozov and Solov'ev 1966, Hamada 1959, 1962). Let (x^1, x^2, x^3) be such a system and as usual the upper suffixes will denote contravariant quantities and the lower suffixes covariant quantities.

The field line equations and the relation $\nabla\cdot\boldsymbol{B}=0$ when expressed in this general coordinate system (Weinberg 1972) become

$$\frac{\mathrm{d}x^1}{B^1}=\frac{\mathrm{d}x^2}{B^2}=\frac{\mathrm{d}x^3}{B^3}$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}B^i)=0$$

where g is the determinant of the covariant metric tensor g_{ij}. Flux conservation through a tube oriented along the x^3 direction yields

$$J=\sqrt{g}B^3\equiv\frac{\partial(q,p)}{\partial(x^1,x^2)}. \tag{14}$$

This reduces to (6) if we remember that in an orthogonal system the covariant components V_i and the ordinary components $\bar{V}_i$ of an arbitrary vector are related by the formula

$$V_i=h_i\bar{V}_i=h_i^2V^i$$

and the covariant metric tensor reduces to

$$g_{ij}=h_i^2\delta_{ij}\qquad \text{(no sum)}.$$

The equations corresponding to $(9a, b)$ now become

$$\{q,H\}_{x^1,x^2}=\sqrt{g}B^i\frac{\partial}{\partial x^i}q \tag{15a}$$

$$\{p,H\}_{x^1,x^2}=\sqrt{g}B^i\frac{\partial}{\partial x^i}p. \tag{15b}$$

Equations $(15a, b)$ together with (14) now determine the Hamiltonian structure of the field configuration.

6. Conclusion

We have shown that it is possible to cast the equations for magnetic field lines into a Hamiltonian form in a very general way through a mappping of the coordinates. By demanding that the conservation laws are fulfilled in the mapped space, we have tied down the Jacobian of the mapping. We have also given an analysis of various canonically equivalent Hamiltonian forms and finally the problem has been solved in a general non-orthogonal coordinate system.

Acknowledgment

We thank Professor B Dasgupta for his valuable comments and also for bringing to our notice the work of Channel.

References

Arnold V I 1979 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
Arnold V I and Avez A 1968 *Ergodic Problems in Classical Mechanics* (New York: Benjamin)
Boozer A H 1983 *Phys. Fluids* **26** 1288
—— 1985a *Princeton Plasma Phys. Lab. Rep. No 2198*
—— 1985b *Princeton Plasma Phys. Lab. Rep. No 2094 R*
Carey J R and Littlejohn R G 1983 *Ann. Phys., NY* **151** 1
Channel P J 1984 *Los Alamos Nat. Lab. Rep. No LA 10048 MS*
Freis R P, Hartman C W, Hamzeh F M and Lichtenberg A J 1973 *Nucl. Fusion* **13** 533
Goldstein 1980 *Classical Mechanics* 2nd edn (New York: Addison-Wesley)
Hamada S 1959 *Prog. Theor. Phys. (Kyoto)* **2** 145
—— 1962 *Nucl. Fusion* **2** 23
Henon M 1983 *Les Houches Session XXXVI* (Amsterdam: North-Holland)
Lichtenberg A J and Lieberman M A 1983 *Regular and Stochastic Motion* (Berlin: Springer)
Morozov A I and Solov'ev L S 1966 *Rev. Plasma Phys.* **2** 58
Miyamoto K 1980 *Plasma Physics for Nuclear Fusion* (Cambridge, MA: MIT Press)
Rechester A B, Rosenbluth M N and White R B 1979 *Intrinsic Stochasticity in Plasmas* ed G Laval and D Cresillon (Orsay: Editions de Physique) p 239
Solov'ev L S and Shafranov V D 1970 *Rev. Plasma Phys.* **5** 1
Weinberg S 1972 *Gravitation and Cosmology* (New York: Wiley)